BUTP-96/5

Implications of $\eta\eta'$ mixing for the decay $\eta \rightarrow 3\pi$

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January 1996

Abstract

Taken by itself, the interference with the η' appears to strongly affect the amplitude of the transition $\eta \rightarrow 3\pi$. I point out that this effect is fictitious and also occurs in the mass spectrum of the pseudoscalars. Chiral symmetry implies that the same combination of effective coupling constants which determines the small deviations from the Gell-Mann-Okubo formula also specifies the symmetry breaking effects in the decay amplitude and thus ensures that these are small.

Work supported in part by Schweizerischer Nationalfonds

The decay $\eta \to 3\pi$ is of particular interest, because it violates isospin symmetry. The electromagnetic interaction is known to produce only very small corrections [1]. Disregarding these, the transition amplitude is proportional to $m_d - m_u$ and thus represents a sensitive probe of the symmetry breaking generated by the quark masses.

Since the quark mass term $\overline{q}mq = m_u\overline{u}u + m_d\,\overline{d}d + m_s\overline{s}s$ breaks SU(3), it generates transitions between the octet and the singlet of pseudoscalar mesons. The consequences for the transition amplitude are discussed in the literature, but the results are contradictory: While a direct evaluation of the mixing effects [2] leads to the conclusion that the current algebra prediction is modified drastically, the chiral perturbation theory calculation to one loop [3] yields the opposite result. The purpose of the present paper is to resolve this paradox.

The direct calculation is based on the effective Lagrangian which describes the low energy structure of QCD in terms of a simultaneous expansion in powers of $1/N_c$, powers of the momenta p and powers of the quark mass matrix m. For a discussion of this framework to first nonleading order and references to the literature, see ref.[4]. The relevant effective field $U(x)$ is an element of U(3) and includes the degrees of freedom of both the pseudoscalar octet and the singlet. The latter is described by the phase of the determinant, $\det U = e^{i\phi_0}$. Counting the three expansion parameters as small quantities of order $1/N_c = O(\delta)$, $p = O(\sqrt{\delta})$, $m = O(\delta)$, the expansion starts with a contribution of order one, given by

$$\mathcal{L}_{\mathit{eff}} = \tfrac{1}{4}F^2\langle\partial_\mu U^\dagger\partial^\mu U\rangle + \tfrac{1}{2}F^2B\langle mU^\dagger + mU\rangle - \tfrac{1}{2}\tau\phi_0^2 + O(\delta)\ , \qquad (1)$$

where matrix traces are abbreviated with the symbol $\langle\ldots\rangle$. The expression involves three effective coupling constants: the pion decay constant $F = O(\sqrt{N_c})$, the constant $B = O(1)$ which determines the magnitude of the quark condensate and the topological susceptibility $\tau = O(1)$. The relative magnitude of the three leading contributions depends on the relative magnitude of the expansion parameters: the first involves two powers of momentum, the second is proportional to the quark mass matrix and the coefficient τ of the third is smaller than F^2 or F^2B by one power of $1/N_c$.

Setting $U = \exp i\varphi/F$, the singlet field is given by the trace $\phi_0 = \langle\varphi\rangle/F$, so that the terms quadratic in φ are $\frac{1}{4}\langle\partial_\mu\varphi\partial^\mu\varphi\rangle - \frac{1}{2}B\langle m\varphi^2\rangle - \frac{1}{2}\tau\langle\varphi\rangle^2/F^2$. For those fields which carry electric charge or strangeness, this expression is

diagonal and leads to the standard current algebra mass formulae,

$$M^2_{\pi^+} = (m_u + m_d)B \quad , \quad M^2_{K^+} = (m_u + m_s)B \quad , \quad M^2_{K^0} = (m_d + m_s)B \quad .$$

The states π^0, η and η' undergo mixing. The mixing matrix is an element of O(3) and may thus be represented in terms of three angles, $\theta_{\eta\eta'}, \epsilon, \epsilon'$. The first arises from the mass difference between the strange and nonstrange quarks and breaks SU(3), while the other two are isospin breaking effects, driven by $m_d - m_u$. To first order in isospin breaking, the relation between the neutral components of the field, $\varphi = \varphi_3\lambda_3 + \varphi_8\lambda_8 + \varphi_9\sqrt{\frac{2}{3}}$ and the mass eigenstates π^0, η, η' is of the form

$$\begin{aligned} \varphi_3 &= \pi^0 - \epsilon\,\eta - \epsilon'\,\eta' \\ \varphi_8 &= \quad \cos\theta_{\eta\eta'}\,(\eta + \epsilon\,\pi^0) + \sin\theta_{\eta\eta'}\,(\eta' + \epsilon'\,\pi^0) \\ \varphi_9 &= -\sin\theta_{\eta\eta'}\,(\eta + \epsilon\,\pi^0) + \cos\theta_{\eta\eta'}\,(\eta' + \epsilon'\,\pi^0) \quad . \end{aligned} \tag{2}$$

In the eigenvalues, the isospin breaking effects are of order $(m_d - m_u)^2$. Neglecting these, the π^0 is degenerate with $\pi^\pm$, $M^2_{\pi^0} = (m_u + m_d)B$. The remaining two eigenvalues involve a new scale, set by the topological susceptibility. Eliminating it, the diagonalization leads to two independent relations among the three quantities $M_\eta, M_{\eta'}, \theta_{\eta\eta'}$, e.g.

$$\sin 2\theta_{\eta\eta'} = -\tfrac{4}{3}\sqrt{2}\,\frac{M_K^2 - M_\pi^2}{M_{\eta'}^2 - M_\eta^2} \quad , \tag{3}$$

$$M_\eta^2 = \tfrac{1}{3}(4M_K^2 - M_\pi^2) + \tfrac{2}{3}\sqrt{2}\,\mathrm{tg}\,\theta_{\eta\eta'}\,(M_K^2 - M_\pi^2) \quad . \tag{4}$$

The isospin breaking angles ϵ, ϵ' are proportional to the quark mass ratio

$$\epsilon_0 \equiv \frac{\sqrt{3}}{4}\,\frac{(m_d - m_u)}{(m_s - \hat{m})} \quad , \qquad \hat{m} = \tfrac{1}{2}(m_u + m_d) \quad .$$

The coefficients of proportionality may be expressed in terms of $\theta_{\eta\eta'}$,

$$\begin{aligned} \epsilon &= \epsilon_0\,\cos\,\theta_{\eta\eta'}\,\frac{\cos\,\theta_{\eta\eta'} - \sin\,\theta_{\eta\eta'}\,\sqrt{2}}{\cos\,\theta_{\eta\eta'} + \sin\,\theta_{\eta\eta'}/\sqrt{2}} \\ \epsilon' &= -2\,\epsilon_0\,\sin\,\theta_{\eta\eta'}\,\frac{\cos\,\theta_{\eta\eta'} + \sin\,\theta_{\eta\eta'}/\sqrt{2}}{\cos\,\theta_{\eta\eta'} - \sin\,\theta_{\eta\eta'}\,\sqrt{2}} \quad . \end{aligned} \tag{5}$$

Note that, in the counting of powers introduced above, $M_\pi^2, M_K^2, M_\eta^2, M_{\eta'}^2$ are treated as small quantities of order δ. According to eq.(3), the mixing angle $\theta_{\eta'\eta}$ is given by a ratio thereof and thus represents a quantity of order one. In this sense, the above formulae are valid to all orders in $\theta_{\eta'\eta}$. Numerically, the ratio $M_{\eta'}^2/M_\eta^2$ is about equal to 3, indicating that the breaking of $\mathrm{U}(3)_R\times\mathrm{U}(3)_L$ generated by the anomaly is larger than the breaking due to m_s, by roughly this factor. The topological susceptibility, which describes the effects of the anomaly in the framework of the effective Lagrangian and is of order $N_c^{\,0}$, is more important than the terms of order mN_c, which account for the symmetry breaking generated by the quark masses.

Consider now the decay $\eta\rightarrow\pi^+\pi^-\pi^0$. To calculate the corresponding transition amplitude with the effective Lagrangian in eq.(1), the expansion in powers of the field φ is needed to order φ^4. The first term yields a contribution proportional to $\langle[\partial_\mu\varphi,\varphi][\partial^\mu\varphi,\varphi]\rangle$. Since the remainder does not involve derivatives, the decay amplitude A involves at most two powers of momentum. Lorentz invariance and crossing symmetry then imply that A is of the form $a+b\,s$, where $s=(p_{\pi^+}+p_{\pi^-})^2$ is the square of the center of mass energy of the charged pion pair. Performing the change of basis (2), one finds that b is given by $-\epsilon/F^2$, where ϵ is one of the mixing angles introduced above. The amplitude may thus be written as

$$A=-\epsilon\,\frac{1}{F^2}(s-s_A)\ \ .$$

The result is of the same structure as the current algebra prediction [5],

$$A=-\epsilon_0\,\frac{1}{F^2}(s-\tfrac{4}{3}M_\pi^2)\ \ .$$

As a consequence of the interference with the η', the quark mass ratio ϵ_0 is replaced by the mixing angle ϵ given in eq.(5). There is a corresponding change also in the value of the constant term, $\frac{4}{3}M_\pi^2\rightarrow s_A$, but this term is inessential, for the following reason. In the limit $m_u,m_d\rightarrow 0$, the amplitude contains two Adler zeros, one at $p_{\pi^+}=0$, the other at $p_{\pi^-}=0$. For the above expression to have this property, the constant s_A must tend to zero if m_u,m_d are turned off. Hence an explicit evaluation would yield a result for s_A proportional to M_π^2: The interference with the η' merely generates an SU(3) correction in the value $\frac{4}{3}$ of the coefficient. Since contributions of order M_π^2 amount to small corrections, I will drop these in the following.

Using the observed values of $M_\pi^2, M_K^2, M_{\eta'}^2 - M_\eta^2$ as an input, the relation (3) yields $\theta_{\eta\eta'} \simeq -22^\circ$, in reasonable agreement with what is found phenomenologically [6]. Inserting this number, formula (5) gives $\epsilon \simeq 2\,\epsilon_0$. So, the net result of the above calculation is that mixing with the η' increases the current algebra prediction for the amplitude by a factor of 2.

I now compare this finding with the one loop result of chiral perturbation theory [3]. This calculation accounts for all effects of first nonleading order, in particular also for those due to $\eta\eta'$ mixing. It is based on $\mathrm{SU}(3)_R \times \mathrm{SU}(3)_L$ and hence only involves the degrees of freedom of the pseudoscalar octet. In this framework, the η' only manifests itself indirectly, through its contributions to the effective coupling constants, like all other states which remain massive in the chiral limit, e.g. the ρ.

Normalizing the amplitude with the kaon mass difference (e.m. self energies removed) and with the pion matrix element of the axial current,[1] the result is of the form [7]

$$A = -\frac{(M_{K^0}^2 - M_{K^+}^2)}{3\sqrt{3}\,F_\pi^2}\,M(s,t,u) \quad , \tag{6}$$

where $M(s,t,u)$ is a lengthy expression, which contains contributions generated by the final state interaction, as well as symmetry breaking terms involving the effective coupling constants L_5, L_7, L_8. The η' hides in the coupling constant L_7, which also occurs if the mass of the η is calculated within the same framework. The explicit expression for the function $M(s,t,u)$ contains this constant through a correction term which is proportional to the deviation from the Gell-Mann-Okubo formula and is denoted by

$$\Delta_{\mathrm{GMO}} \equiv \frac{4M_K^2 - 3M_\eta^2 - M_\pi^2}{M_\eta^2 - M_\pi^2} \quad .$$

Dropping all other terms and disregarding contributions of order M_π^2, the one loop result reduces to

$$M(s,t,u) = \frac{3\,s}{M_\eta^2 - M_\pi^2}(1 + \tfrac{2}{3}\Delta_{\mathrm{GMO}}) + \ldots = \frac{9\,s}{4(M_K^2 - M_\pi^2)}(1 + \Delta_{\mathrm{GMO}}) + \ldots \quad ,$$

[1] Since the expression accounts for the corrections of order m, one needs to distinguish between the constant F in the effective Lagrangian and the observed decay constants F_π, F_K, which differ from F through contributions of order m.

where I have used the identity $(M_\eta^2 - M_\pi^2)(1 + \frac{1}{3}\Delta_{\rm GMO}) = \frac{4}{3}(M_K^2 - M_\pi^2)$. The current algebra mass formulae quoted above show that, at leading order of the chiral perturbation series, the ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$ is given by $(m_d - m_u)/(m_s - \hat{m}) = 4\epsilon_0/\sqrt{3}$. Since the corresponding first order corrections do not involve the coupling constant L_7, they are irrelevant in the present context and the same also holds for the difference between F_π and F.

With these simplifications, eq.(6) reduces to $A = -(\epsilon_0/F^2)\, s\,(1 + \Delta_{\rm GMO})$: Up to small corrections of order M_π^2, the contribution from the symmetry breaking terms amounts to an overall renormalization of the amplitude, $\epsilon_0 \rightarrow \epsilon_0\,(1 + \Delta_{\rm GMO})$. The experimental value of the deviation from the Gell-Mann-Okubo formula, $\Delta_{\rm GMO} = 0.22$, shows that the modification is of reasonable size, confirming the general rule of thumb, according to which first order SU(3) breaking effects are typically of order 25%. The second order contributions, which the one loop formula neglects, are expected to be of the order of the square of this. Clearly, the outcome of the calculation described earlier is in flat contradiction with chiral perturbation theory.

To identify the origin of the disagreement, I return to the earlier calculation and express the angular factor occurring in eq.(5) in terms of the masses of the particles. Solving the relation (4) for $\mathrm{tg}\,\theta_{\eta\eta'}$ and inserting the result, the angular factor becomes

$$\frac{\cos\theta_{\eta'\eta} - \sin\theta_{\eta'\eta}\sqrt{2}}{\cos\theta_{\eta'\eta} + \sin\theta_{\eta'\eta}/\sqrt{2}} = \frac{2(2M_K^2 - M_\eta^2 - M_\pi^2)}{M_\eta^2 - M_\pi^2} \equiv 1 + \Delta_{\rm GMO} \quad .$$

This is remarkable, because it shows that the expression for the mixing angle ϵ may equally well be written as

$$\epsilon = \epsilon_0\,\{1 + \Delta_{\rm GMO}\}\cos\theta_{\eta'\eta} \quad . \tag{7}$$

In this form, the result of the direct calculation differs from the corresponding term in the one loop prediction of chiral perturbation theory only by a factor of $\cos\theta_{\eta'\eta} \simeq 0.93$, which represents a correction of order $(m_s - \hat{m})^2$ and is beyond the accuracy of the one loop result. I conclude that the two calculations are consistent with one another. In particular, it is incorrect to amalgamate the two by multiplying the one loop formula with the enhancement factor occurring in eq.(5).

The above expression for the angular factor shows that the result of the direct calculation is subject to an uncertainty comparable to the effect itself:

Depending on whether one uses eq.(3) or eq.(4), i.e. takes the observed values of $M_{\eta'}^2 - M_\eta^2$ or M_η^2 as an input, the calculation yields $\epsilon \simeq 2\,\epsilon_0$ or $\epsilon \simeq 1.2\,\epsilon_0$, respectively. The problem arises from the fact that the mass formula (4) is not in good agreement with observation. If the second term is omitted, the relation reduces to the Gell-Mann-Okubo formula, which predicts $M_\eta \simeq 566\,\mathrm{MeV}$, slightly larger than what is observed. The second term indeed lowers the result, but the shift is much too large: While the Gell-Mann-Okubo prediction for M_η^2 only differs from the experimental value by 7 %, the repulsion generated by mixing now yields a number which is too low by about 20 %. As pointed out in ref.[7], early determinations of the mixing angle failed for precisely this reason: These were based on the assumption that the observed deviation from the Gell-Mann-Okubo formula is exclusively due to $\eta\eta'$ mixing and thus underestimated the magnitude of $\theta_{\eta'\eta}$ by about a factor of two.

The above discrepancies do not indicate that the expansion of the effective Lagrangian in powers of $1/N_c, p$ and m fails. Deviations of this order of magnitude are to be expected within a framework which only considers the leading term of the expansion. The effective Lagrangian in eq.(1) also predicts that F_K is equal to F_π, while, experimentally, the two quantities differ by the factor 1.22. There is no reason why in the case of the masses, the corrections generated by the higher order contributions of the expansion should be smaller.

The observed mass pattern is perfectly consistent with the assumption that the terms neglected in eq.(1) are small, but they definitely are different from zero. The main point here is that the same terms necessarily also affect the amplitude of the transition $\eta \to 3\pi$. The result of the chiral perturbation theory calculation amounts to a low energy theorem: To order p^4, the slope of the decay amplitude involves the same combination of couplings which determines the deviation from the Gell-Mann-Okubo formula. While the effective Lagrangian in eq.(1) only accounts for the coupling constant L_7, which is related to $\eta\eta'$ mixing, the one loop result receives significant contributions also from L_5 and L_8. In the framework of pole models [8], these couplings are dominated by the exchange of scalar particles. Indeed, the particle data table shows that the mass of the lightest scalars is comparable with $M_{\eta'}$. These particles do not undergo mixing with pseudoscalar one-particle states, but with the ground state as well as with two-particle states. The corresponding effect in the square of the pseudoscalar masses is

also of order m^2 and is of opposite sign. It is suppressed by a relatively large energy denominator, because the masses of the scalars are large compared to those of the pseudoscalar octet, but the energy denominator is essentially the same as the one which suppresses the shift generated by $\eta\eta'$ mixing, so that the effects are of the same order of magnitude.

This explains why the direct calculation does not yield a decent estimate for ϵ, unless the result is written in the form (7), where it differs from the chiral perturbation theory result only by a factor of $\cos\theta_{\eta'\eta}$. As noted above, this factor represents a second order correction of typical size. I conclude that there is no indication for the symmetry breaking effects of higher order to be unusually large – the factor $\cos\theta_{\eta'\eta}$ may be taken as an estimate for the uncertainties in the decay amplitude due to these.

The current algebra prediction for the decay $\eta \to 3\pi$ also receives corrections from a quite different source: final state interactions. These are generated predominantly by two-particle branch cuts and are responsible for the bulk of the one loop corrections. The corresponding higher order contributions may be worked out from unitarity, using dispersion relations and relying on chiral perturbation theory only to determine the subtraction constants [9, 10]. Viewed in this perspective, the above discussion implies that the one loop predictions for the subtraction constants are trustworthy: These account for all symmetry breaking effects of first nonleading order, in particular for those due to $\eta\eta'$ mixing, and there is no indication for large corrections from higher orders.

Note that these statements need not hold for radiative transitions, which may well be distorted by the pole due to η'-exchange [2, 11]. The 3π channel is special, because the transition amplitude is determined by the effective Lagrangian of the strong interaction — this is why it is firmly tied to the mass spectrum of the pseudoscalars.

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